DAY NINE

Unit Test 1

(Algebra)

1 If $f(x) = \begin{vmatrix} \cos x & x & 1 \\ 2\sin x & x^2 & 2x \\ \tan x & x & 1 \end{vmatrix}$, then $\lim_{x \to 0} \frac{f'(x)}{x}$ is equal to

(a) 1 (b) −1 (c) 2 (d) −2

2 If $\log_{0.5}(x-1) < \log_{0.25}(x-1)$, then x lies in the interval

(a) $(2, \infty)$ (b) $(3, \infty)$ (c) $(-\infty, 0)$ (d) $(0, 3)$

3 Sum of n terms of series $12 + 16 + 24 + 40 + \dots$ will be

(a) $2(2^n - 1) + 8n$
(b) $2(2^n - 1) + 6n$
(c) $3(2^n - 1) + 8n$
(d) $4(2^n - 1) + 8n$

4 Let a, b and $c \in R$ and $a \neq 0$. If α is a root of $a^2x^2 + bx + c = 0$, β is a root of $a^2x^2 - bx - c = 0$ and $0 < \alpha < \beta$, then the equation of $a^2x^2 + 2bx + 2c = 0$ has a root γ, that always satisfies

(a) $\gamma = \alpha$ (b) $\gamma = \beta$
(c) $\gamma = (\alpha + \beta)/2$ (d) $\alpha < \gamma < \beta$

5 Between two numbers whose sum is $2\frac{1}{6}$ an even number of arithmetic means are inserted. If the sum of these means exceeds their number by unity, then the number of means are

(a) 12 (b) 10
(c) 8 (d) None of these

6 If $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, then which of the following is not true?

(a) $\lim_{n \to \infty} \frac{1}{n^2} A^{-n} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$ (b) $\lim_{n \to \infty} \frac{1}{n} A^{-n} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$

(c) $A^{-n} = \begin{bmatrix} 1 & 0 \\ -n & 1 \end{bmatrix}, \forall n \in N$ (d) None of these

7 From 50 students taking examinations in Mathematics, Physics and Chemistry, 37 passed Mathematics, 24 Physics and 43 Chemistry. Atmost 19 passed Mathematics and Physics, atmost 29 passed Mathematics and Chemistry and atmost 20 passed Physics and Chemistry. The largest possible number that could have passed all three examinations is

(a) 11 (b) 12 (c) 13 (d) 14

8 The inequality $|z - 4| < |z - 2|$ represents the region given by

(a) $\text{Re}(z) > 0$ (b) $\text{Re}(z) < 0$
(c) $\text{Re}(z) > 3$ (d) None of these

9 If 1, ω and ω^2 be the three cube roots of unity, then $(1 + \omega)(1 + \omega^2)(1 + \omega^4)\dots 2n$ factors is equal to

(a) 1 (b) −1
(c) 0 (d) None of these

10 If $a < 0$, then the positive root of the equation $x^2 - 2a|x - a| - 3a^2 = 0$ is

(a) $a(-1 - \sqrt{6})$ (b) $a(1 - \sqrt{2})$
(c) $a(1 - \sqrt{6})$ (d) $a(1 + \sqrt{2})$

11 The common roots of the equations $z^3 + 2z^2 + 2z + 1 = 0$ and $z^{1985} + z^{100} + 1 = 0$ are

(a) $-1, \omega$ (b) $-1, \omega^2$
(c) ω, ω^2 (d) None of these

12 Let z_1, z_2 and z_3 be three points on $|z| = 1$. If θ_1, θ_2 and θ_3 are the arguments of z_1, z_2 and z_3 respectively, then $\cos(\theta_1 - \theta_2) + \cos(\theta_2 - \theta_3) + \cos(\theta_3 - \theta_1)$

(a) $\geq \frac{3}{2}$ (b) $\geq -\frac{3}{2}$

(c) $\leq \frac{-3}{2}$ (d) None of these

13 If the roots of the equation

$$(a^2+b^2)x^2+2(bc+ad)x+(c^2+d^2)=0$$

are real, then a^2, bd and c^2 are in

(a) AP (b) GP
(c) HP (d) None of these

14 150 workers were engaged to finish a piece of work in a certain number of days. Four workers dropped the second day, four more workers dropped the third day and so on. It takes 8 more days to finish the work now. Then, the number of days in which the work was completed is

(a) 29 days (b) 24 days (c) 25 days (d) 26 days

15 Let R be a relation defined by $R=\{(x, x^3): x$ is a prime number $<10\}$, then which of the following is true?

(a) $R=\{(1,1),(2,8),(3,27),(4,64),(5,125),(6,216),(7,343),(8,512),(9,729)\}$
(b) $R=\{(2,8),(3,27),(5,125),(7,343)\}$
(c) $R=\{(2,8),(4,64),(6,216),(8,512)\}$
(d) None of the above

16 If $b>a$, then the equation $(x-a)(x-b)-1=0$, has

(a) both the roots in $[a, b]$
(b) both the roots in $(-\infty, a]$
(c) both the roots in (b, ∞)
(d) one root in $(-\infty, a)$ and other in (b, ∞)

17 The value of x satisfying $\log_2(3x-2)=\log_{1/2} x$ is

(a) $-\frac{1}{3}$ (b) 2
(c) $\frac{1}{2}$ (d) None of these

18 If $f(x, n)=\sum_{r=1}^{n} \log_x\left(\frac{r}{x}\right)$, then the value of x satisfying the equation $f(x, 11)=f(x, 12)$ is

(a) 10 (b) 11
(c) 12 (d) None of these

19 The three numbers a, b and c between 2 and 18 are such that their sum is 25, the numbers 2, a and b are consecutive terms of an AP and the numbers b, c and 18 are consecutive terms of a GP. The three numbers are

(a) 3, 8, 14 (b) 2, 9, 14
(c) 5, 8, 12 (d) None of these

20 If X is the set of all complex numbers z such that $|z|=1$, then the relation R defined on X by

$|\arg z_1-\arg z_2|=\frac{2\pi}{3}$, is

(a) reflexive (b) symmetric
(c) transitive (d) anti-symmetric

21 If α and β are the roots of the equation $ax^2-2bx+c=0$, then $\alpha^3\beta^3+\alpha^2\beta^3+\alpha^3\beta^2$ is equal to

(a) $\frac{c^2}{a^3}(c-2b)$ (b) $\frac{c^2}{a^3}(c+2b)$
(c) $\frac{bc^2}{a^3}$ (d) None of these

22 If the roots of the equation $x^2-2ax+a^2+a-3=0$ are real and less than 3, then

(a) $a<2$ (b) $2\le a\le 3$ (c) $3\le a\le 4$ (d) $a>4$

23 The integer k for which the inequality $x^2-2(4k-1)x+15k^2-2k-7>0$ is valid for any x, is

(a) 2 (b) 3
(c) 4 (d) None of these

24 The maximum sum of the series

$20+19\frac{1}{3}+18\frac{2}{3}+18+\ldots$ is

(a) 310 (b) 290
(c) 320 (d) None of these

25 The number of common terms to the two sequences 17, 21, 25, ..., 417 and 16, 21, 26, ..., 466 is

(a) 21 (b) 19 (c) 20 (d) 91

26 If the sum of the first three terms of a GP is 21 and the sum of the next three terms is 168, then the first term and the common ratio is

(a) 3, 4 (b) 2, 4
(c) 3, 2 (d) None of these

27 The sum to n terms of the series

$$\frac{1}{1+1^2+1^4}+\frac{2}{1+2^2+2^4}+\frac{3}{1+3^2+3^4}+\ldots, \text{ is}$$

(a) $\frac{n^2+n}{2(n^2+n+1)}$ (b) $\frac{n^2-n}{2(n^2+n+1)}$
(c) $\frac{n^2+n}{2(n^2-n+1)}$ (d) None of these

28 If C is a skew-symmetric matrix of order n and X is $n\times 1$ column matrix, then $X'CX$ is a

(a) scalar matrix (b) unit matrix
(c) null matrix (d) None of these

29 Which of the following is correct?

(a) Skew-symmetric matrix of an even order is always singular
(b) Skew-symmetric matrix of an odd order is non-singular
(c) Skew-symmetric matrix of an odd order is singular
(d) None of the above

30 If $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=5$, then the value of

$$\Delta=\begin{vmatrix} b_2c_3-b_3c_2 & a_3c_2-a_2c_3 & a_2b_3-a_3b_2 \\ b_3c_1-b_1c_3 & a_1c_3-a_3c_1 & a_3b_1-a_1b_3 \\ b_1c_2-b_2c_1 & a_2c_1-a_1c_2 & a_1b_2-a_2b_1 \end{vmatrix} \text{ is}$$

(a) 5 (b) 25 (c) 125 (d) 0

31 The number of seven letter words that can be formed by using the letters of the word 'SUCCESS' so that the two C are together but no two S are together, is

(a) 24 (b) 36
(c) 54 (d) None of these

32 The greatest integer less than or equal to $(\sqrt{2}+1)^6$ is
(a) 196 (b) 197
(c) 198 (d) 199

33 If A and B are square matrices such that $B = -A^{-1}BA$, then
(a) $AB + BA = O$
(b) $(A+B)^2 = A^2 - B^2$
(c) $(A+B)^2 = A^2 + 2AB + B^2$
(d) $(A+B)^2 = A + B$

34 The determinant $\begin{vmatrix} xp+y & x & y \\ yp+z & y & z \\ 0 & xp+y & yp+z \end{vmatrix} = 0$, if
(a) x, y, z are in AP (b) x, y, z are in GP
(c) x, y, z are in HP (d) xy, yz, zx are in AP

35 The value of k, for which the system of equations $x + ky + 3z = 0, 3x + ky - 2z = 0$ and $2x + 3y - 4z = 0$ possess a non-trivial solution over the set of rationals, is
(a) $-\frac{33}{2}$ (b) $\frac{33}{2}$
(c) 11 (d) None of these

36 The number of groups that can be made from 5 different green balls, 4 different blue balls and 3 different red balls, if atleast 1 green and 1 blue ball is to be included is
(a) 3700 (b) 3720
(c) 4340 (d) None of these

37 If n is an integer greater than 1, then $a - {}^nC_1(a-1) + {}^nC_2(a-2) + \ldots + (-1)^n(a-n)$ is equal to
(a) a (b) 0 (c) a^2 (d) 2^n

38 If α, β and γ are the roots of the equation $x^2(x+e) = e(x+1)$. Then, the value of the determinant $\begin{vmatrix} 1+\alpha & 1 & 1 \\ 1 & 1+\beta & 1 \\ 1 & 1 & 1+\gamma \end{vmatrix}$ is
(a) -1 (b) 1
(c) 0 (d) $1 + \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

39 For all natural number $n > 1$, $2^{4n} - 15n - 1$ is divisible by
(a) 225 (b) 125
(c) 325 (d) None of these

40 If x is so small that its two and higher power can be neglected and if $(1-2x)^{-1/2}(1-4x)^{-5/2} = 1 + kx$, then k is equal to
(a) -2 (b) 1
(c) 10 (d) 11

41 If $x = -5$ is a root of $\begin{vmatrix} 2x+1 & 4 & 8 \\ 2 & 2x & 2 \\ 7 & 6 & 2x \end{vmatrix} = 0$, then the other two roots are
(a) 3, 3.5 (b) 1, 3.5
(c) 3, 6 (d) 2, 6

42 The 8th term of $\left(3x + \frac{2}{3x^2}\right)^{12}$ when expanded in ascending power of x, is
(a) $\frac{228096}{x^3}$ (b) $\frac{228096}{x^9}$
(c) $\frac{328179}{x^9}$ (d) None of these

43 The greatest term in the expansion of $(3-5x)^{11}$ when $x = \frac{1}{5}$, is
(a) 55×3^9 (b) 46×3^9
(c) 55×3^6 (d) None of these

44 In an examination a candidate has to pass in each of the papers to be successful. If the total number of ways to fail is 63, how many papers are there in the examination?
(a) 6 (b) 8 (c) 10 (d) 12

45 A is a set containing n elements. A subset P of A is chosen. The set A is reconstructed by replacing the elements of P. A subset Q of A is again chosen. The number of ways of choosing P and Q, so that $P \cap Q$ contains exactly two elements is
(a) $9 \cdot {}^nC_2$ (b) $3^n - {}^nC_2$
(c) $2 \cdot {}^nC_n$ (d) None of these

46 If the sets A and B are defined as
$$A = \{(x, y) : y = \frac{1}{x}, 0 \neq x \in R\}$$
and $B = \{(x, y) : y = -x, x \in R\}$, then
(a) $A \cap B = A$ (b) $A \cap B = B$
(c) $A \cap B = \phi$ (d) None of these

47 There are 16 points in a plane no three of which are in a straight line except 8 which are all in a straight line. The number of triangles can be formed by joining them equals
(a) 1120 (b) 560 (c) 552 (d) 504

48 The value of the natural numbers n such that inequality $2^n > 2n + 1$ is valid, is
(a) for $n \geq 3$ (b) for $n < 3$ (c) for mn (d) for any n

49 Let $w = \cos\left(\frac{2\pi}{7}\right) + i \sin\left(\frac{2\pi}{7}\right)$ and $\alpha = w + w^2 + w^4$ and $\beta = w^3 + w^5 + w^6$, then $\alpha + \beta$ is equal to
(a) 0 (b) -1 (c) -2 (d) 1

50 Let $H(x) = \frac{f(x)}{g(x)}$, where $f(x) = 1 - 2\sin^2 x$ and $g(x) = \cos 2x, \forall f : R \to [-1, 1]$ and $g : R \to [-1, 1]$. Domain and range of $H(x)$ are respectively
(a) R and $\{1\}$
(b) R and $\{0, 1\}$
(c) $R \sim \{(2n+1)\frac{\pi}{4}\}$ and $\{1\}, n \in I$
(d) $R \sim \left\{(2n+1)\frac{\pi}{2}\right\}$ and $\{0, 1\}, n \in I$

51 Let T_n denote the number of triangles which can be formed using the vertices of a regular polygon of n sides. If $T_{n+1} - T_n = 21$, then n equals

(a) 4 (b) 6
(c) 7 (d) None of these

52 If r is a real number such that $|r| < 1$ and if $a = 5(1 - r)$, then

(a) $0 < a < 5$ (b) $-5 < a < 5$
(c) $0 < a < 10$ (d) $0 \le a < 10$

Directions (Q. Nos. 53-57) *Each of these questions contains two statements : Statement* I (*Assertion*) *and Statement* II (*Reason*). *Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes* (*a*), (*b*), (*c*) *and* (*d*) *given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true; Statement II is false
(d) Statement I is false; Statement II is true

53 **Statement I** The number of natural numbers which divide 10^{2009} but not 10^{2008} is 4019.

Statement II If p is a prime, then number of divisors of p^n is $p^{n+1} - 1$.

54 Suppose $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. Let X be a 2×2 matrices such that $X'AX = B$.

Statement I X is non-singular and $\det(X) = \pm 2$.

Statement II X is a singular matrix.

55 The general term in the expansion of $(a + x)^n$ is ${}^nC_r\, a^{n-r}\, x^r$.

Statement I The third term in the expansion of $\left(2x + \frac{1}{x^2}\right)^m$ does not contain x. The value of x for which that term equal to the second term in the expansion of $(1 + x^3)^{30}$ is 2.

Statement II $(a + x)^n = \sum_{r=0}^{n} {}^nC_r\, a^{n-r} x^r$.

56 Sets A and B have four and eight elements, respectively.

Statement I The minimum number of elements in $A \cup B$ is 8.

Statement II $A \cap B = 5$

57 Let $a \ne 0, p \ne 0$ and $\Delta = \begin{vmatrix} a & b & c \\ 0 & p & q \\ p & q & 0 \end{vmatrix}$

Statement I If the equations $ax^2 + bx + c = 0$ and $px + q = 0$ have a common root, then $\Delta = 0$.

Statement II If $\Delta = 0$, then the equations $ax^2 + bx + c = 0$ and $px + q = 0$ have a common root.

58 Assume X, Y, Z, W and P are matrices of order $2 \times n$, $3 \times k$, $2 \times p$, $n \times 3$ and $p \times k$, respectively.

Now, consider the following statements

I. $PY + WY$ will be defined for $k = 3$ and $p = n$.

II. The order of the matrix $7X - 5Z$ is $n \times 2$ (if $p = n$).

Choose the correct option.

(a) Only I is true (b) Only II is true
(c) Both I and II are true (d) Neither I nor II is true

ANSWERS

1. (d)	**2.** (a)	**3.** (d)	**4.** (d)	**5.** (a)	**6.** (a)	**7.** (d)	**8.** (c)	**9.** (a)	**10.** (b)
11. (c)	**12.** (b)	**13.** (b)	**14.** (c)	**15.** (b)	**16.** (d)	**17.** (d)	**18.** (c)	**19.** (c)	**20.** (b)
21. (b)	**22.** (a)	**23.** (b)	**24.** (a)	**25.** (c)	**26.** (c)	**27.** (a)	**28.** (c)	**29.** (c)	**30.** (b)
31. (a)	**32.** (b)	**33.** (a)	**34.** (b)	**35.** (b)	**36.** (b)	**37.** (b)	**38.** (c)	**39.** (a)	**40.** (d)
41. (b)	**42.** (a)	**43.** (a)	**44.** (a)	**45.** (d)	**46.** (c)	**47.** (d)	**48.** (a)	**49.** (b)	**50.** (c)
51. (c)	**52.** (c)	**53.** (c)	**54.** (c)	**55.** (b)	**56.** (c)	**57.** (c)	**58.** (a)		

Hints and Explanations

1 Applying $R_1 \to R_1 - R_3$

$$f(x) = \begin{vmatrix} \cos x - \tan x & 0 & 0 \\ 2\sin x & x^2 & 2x \\ \tan x & x & 1 \end{vmatrix}$$

$$= (\cos x - \tan x)(x^2 - 2x^2)$$
$$= -x^2(\cos x - \tan x)$$

$\therefore f'(x) = -2x(\cos x - \tan x) - x^2(-\sin x - \sec^2 x)$

$\therefore \lim\limits_{x\to 0} \dfrac{f'(x)}{x} = \lim\limits_{x\to 0} [-2(\cos x - \tan x) + \lim\limits_{x\to 0} x(\sin x + \sec^2 x)]$

$= -2 \times 1 = -2$

2 Given, $\log_{0.5}(x-1) < \log_{0.25}(x-1)$

$\Rightarrow \log_{0.5}(x-1) < \log_{(0.5)^2}(x-1)$

$\Rightarrow \log_{0.5}(x-1) < \dfrac{1}{2}\log_{0.5}(x-1)$

$\Rightarrow \log_{0.5}(x-1) < 0 \Rightarrow x - 1 > 1$

$\therefore x > 2$

3 Let, $S_n = 12 + 16 + 24 + \ldots + T_n$

Again, $S_n = 12 + 16 + \ldots + T_n$

$0 = (12 + 4 + 8 + 16 + \ldots$ upto n terms$) - T_n$

$\therefore T_n = 12 + \dfrac{4(2^{n-1} - 1)}{2 - 1} = 2^{n+1} + 8$

On putting $n = 1, 2, 3, \ldots$, we get

$T_1 = 2^2 + 8, T_2 = 2^3 + 8, T_3 = 2^4 + 8 \ldots$

$\therefore S_n = T_1 + T_2 + \ldots + T_n$

$= (2^2 + 2^3 + \ldots$ upto n terms$)$ $+ (8 + 8 + \ldots$ upto n terms$)$

$= \dfrac{2^2(2^n - 1)}{2 - 1} + 8n$

$= 4(2^n - 1) + 8n$

4 Let $f(x) = a^2x^2 + 2bx + 2c$

$\because$ We have, $a^2\alpha^2 + b\alpha + c = 0$

and $a^2\beta^2 - b\beta - c = 0$

$\therefore f(\alpha) = a^2\alpha^2 + 2b\alpha + 2c = b\alpha + c$

$= -a^2\alpha^2$

$f(\beta) = a^2\beta^2 + 2b\beta + 2c = 3(b\beta + c)$

$= 3a^2\beta^2$

But $0 < \alpha < \beta \Rightarrow \alpha, \beta$ are real number.

$\therefore f(\alpha) < 0, f(\beta) > 0$

Hence, $\alpha < \gamma < \beta$.

5 Let $2n$ arithmetic means be $A_1, A_2, \ldots, A_{2n}$ between a and b.

Then, $A_1 + A_2 + \ldots + A_{2n} = \dfrac{a+b}{2} \times 2n$

$= \dfrac{13/6}{2} \times 2n = \dfrac{13n}{6}$

Given, $A_1 + A_2 + \ldots + A_{2n} = 2n + 1$

$\Rightarrow 2n + 1 = \dfrac{13n}{6}$

$\Rightarrow 12n + 6 = 13n$

$\therefore n = 6$

Hence, the number of means

$= 2 \times 6 = 12$

6 $A^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

$\Rightarrow (A^{-1})^2 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

Similarly, $(A^{-1})^n = \begin{bmatrix} 1 & 0 \\ -n & 1 \end{bmatrix}$

Now, $\lim\limits_{n\to\infty} \dfrac{1}{n} A^{-n} = \lim\limits_{n\to\infty} \begin{bmatrix} 1/n & 0 \\ -1 & 1/n \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$

and $\lim\limits_{n\to\infty} \dfrac{1}{n^2} A^{-n} = \lim\limits_{n\to\infty} \begin{bmatrix} 1/n^2 & 0 \\ -\dfrac{1}{n} & 1/n^2 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

7 Given, $n(M \cup P \cup C) = 50$,

$n(M) = 37, n(P) = 24, n(C) = 43$

$n(M \cap P) \le 19, n(M \cap C) \le 29$,

$n(P \cap C) \le 20$

$\therefore n(M \cup P \cup C) = n(M) + n(P) + n(C) - n(M \cap P) - n(M \cap C) - n(P \cap C) + n(M \cap P \cap C)$

$\Rightarrow 50 = 37 + 24 + 43 - n(M \cap P) - n(M \cap C) - n(P \cap C) + n(M \cap P \cap C)$

$\Rightarrow n(M \cap P \cap C) = n(M \cap P) + n(M \cap C) + n(P \cap C) - 54$

$\therefore n(M \cap P \cap C)$

$\le 19 + 29 + 20 - 54 = 14$

8 $|(x-4) + iy|^2 < |(x-2) + iy|^2$

[let $z = x + iy$]

$\Rightarrow (x-4)^2 + y^2 < (x-2)^2 + y^2$

$\Rightarrow -4x < -12 \Rightarrow x > 3$

$\therefore \text{Re}(z) > 3$

9 $(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)\ldots$ to $2n$ factors

$= (1+\omega)(1+\omega^2)(1+\omega)(1+\omega^2)\ldots$ to $2n$ factors

$= [(1+\omega)(1+\omega)\ldots$ to n factors$]$ $[(1+\omega^2)(1+\omega^2)\ldots$ to n factors$]$

$= (1+\omega)^n(1+\omega^2)^n$

$= (1 + \omega + \omega^2 + \omega^3)^n$

$= (0 + \omega^3)^n = \omega^{3n} = 1$

10 If $x \ge a$, then $x^2 - 2a(x-a) - 3a^2 = 0$

$\Rightarrow x^2 - 2ax - a^2 = 0$

$\therefore x = \dfrac{2a \pm \sqrt{4a^2 + 4a^2}}{2} = a(1 \pm \sqrt{2})$

Since, $x \ge a$

$\therefore x = a(1 + \sqrt{2})$, it is impossible because $a < 0$

$\therefore x = a(1 - \sqrt{2})$

If $x < a$, then $x^2 + 2a(x-a) - 3a^2 = 0$

$\Rightarrow x^2 + 2ax - 5a^2 = 0$

$\therefore x = (-1 \pm \sqrt{6})a$

[impossible $x < a$ and $a < 0$]

11 $z^3 + 2z^2 + 2z + 1 = 0$

$\Rightarrow (z+1)(z^2 + z + 1) = 0$

$\Rightarrow z = -1, \omega, \omega^2$

But $z = -1$ does not satisfy the second equation.

Hence, common roots are ω and ω^2.

12 We have, $|z_1| = |z_2| = |z_3| = 1$

Now, $|z_1 + z_2 + z_3| \ge 0$

$\Rightarrow |z_1|^2 + |z_2|^2 + |z_3|^2 + 2\,\text{Re}(z_1\bar{z}_2 + z_2\bar{z}_3 + z_3\bar{z}_1) \ge 0$

$\Rightarrow 3 + 2[\cos(\theta_1 - \theta_2) + \cos(\theta_2 - \theta_3) + \cos(\theta_3 - \theta_1)] \ge 0$

$\Rightarrow \cos(\theta_1 - \theta_2) + \cos(\theta_2 - \theta_3) + \cos(\theta_3 - \theta_1) \ge -\dfrac{3}{2}$

13 Here, $D \ge 0$

$\therefore 4(bc + ad)^2 - 4(a^2 + b^2)(c^2 + d^2) \ge 0$

$\Rightarrow b^2c^2 + a^2d^2 + 2abcd - a^2c^2 - a^2d^2 - b^2c^2 - b^2d^2 \ge 0$

$\Rightarrow (ac - bd)^2 \le 0$

$\Rightarrow ac - bd = 0$

[since, square of any expression cannot be negative]

$\therefore b^2d^2 = a^2c^2$

Hence, a^2, bd and c^2 are in GP.

14 Here, $a = 150$ and $d = -4$

$S_n = \dfrac{n}{2}[2 \times 150 + (n-1)(-4)]$

$= n(152 - 2n)$

Had the workers not dropped, then the work would have finished is $(n-8)$ days with 150 workers working on each day.

$\therefore \quad n(152 - 2n) = 150(n - 8)$

$\Rightarrow \quad n^2 - n - 600 = 0$

$\Rightarrow \quad (n - 25)(n + 24) = 0$

$\therefore \quad n = 25$

[since, n cannot be negative]

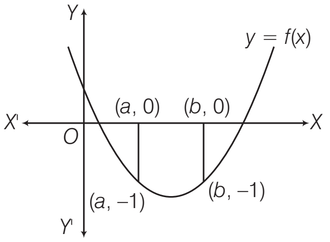
15 Given, x is a prime < 10

$\therefore \quad x = \{2, 3, 5, 7\}$

Now, from $R = \{(x, x^3) : x = 2, 3, 5, 7\}$
$= \{(2, 8), (3, 27), (5, 125), (7, 343)\}$

16 Let $f(x) = (x - a)(x - b) - 1$

We observe that the coefficient of x^2 in $f(x)$ is positive and $f(a) = f(b) = -1$. Thus, the graph of $f(x)$ is as shown in figure given below



It is evident from the graph that one of the roots of $f(x) = 0$ lies in $(-\infty, a)$ and the other root lies in (b, ∞).

17 Given, $\log_2 (3x - 2) = \log_{1/2} x$

$\Rightarrow \quad \log_2 (3x - 2) = -\log_2 x$

$\Rightarrow \quad \log_2 (3x - 2) = \log_2 x^{-1}$

$\Rightarrow \quad 3x - 2 = x^{-1}$

$\Rightarrow \quad 3x^2 - 2x - 1 = 0$

$\Rightarrow \quad (3x + 1)(x - 1) = 0$

$\Rightarrow \quad x = 1 \text{ or } x = -\dfrac{1}{3}$

$\therefore \quad x = 1$

[since, negative of x cannot satisfy the given equation]

18 $f(x, n) = \sum_{r=1}^{n} (\log_x r - \log_x x)$

$= \sum_{r=1}^{n} (\log_x r - 1) = \log_x (1 \cdot 2 \ldots n) - n$

$= \log_x n! - n$

Given, $\quad f(x, 11) = f(x, 12)$

$\Rightarrow \quad \log_x (11!) - 11 = \log_x (12!) - 12$

$\Rightarrow \quad \log_x \left(\dfrac{12!}{11!}\right) = 1 \Rightarrow \log_x (12) = 1$

$\therefore \quad x = 12$

19 Given, $a + b + c = 25$ $\quad \ldots$(i)

Since, $2, a, b$ are in AP, therefore

$2a = 2 + b$ $\quad \ldots$(ii)

Since, $b, c, 18$ are in GP, therefore

$c^2 = 18b$ $\quad \ldots$(iii)

From Eqs. (i) and (ii), we get

$3b = 48 - 2c$

From Eq. (iii), we get

$c^2 = 6(48 - 2c) = 288 - 12c$

$\Rightarrow \quad c^2 + 12c - 288 = 0$

$\Rightarrow \quad (c + 24)(c - 12) = 0$

$\Rightarrow \quad c = 12 \text{ as } c \neq -24$

$\therefore \quad b = 8 \quad \text{and} \quad a = 5$

20 $\because |z| = 1 \Rightarrow z = \cos\theta + i \sin\theta$

$\therefore \arg(z) = \theta$

Let, $\arg(z_1) = \theta_1$ and $\arg(z_2) = \theta_2$

Then, $z_1 R z_2 \Rightarrow |\arg z_1 - \arg z_2| = \dfrac{2\pi}{3}$

$\Rightarrow \quad z_1 R z_2 \Rightarrow |\theta_1 - \theta_2| = \dfrac{2\pi}{3}$

$\Rightarrow \quad |\theta_2 - \theta_1| = \dfrac{2\pi}{3}$

$\Rightarrow \quad z_2 R z_1$

Hence, it is symmetric.

21 Here, $\alpha + \beta = \dfrac{2b}{a}$ and $\alpha\beta = \dfrac{c}{a}$

Now, $(\alpha\beta)^3 + \alpha^2\beta^2(\beta + \alpha)$

$= \left(\dfrac{c}{a}\right)^3 + \dfrac{c^2}{a^2}\left(\dfrac{2b}{a}\right)$

$= \dfrac{c^2(c + 2b)}{a^3}$

22 According to the question,

$D \geq 0$ and $f(3) > 0$

$\Rightarrow \quad 4a^2 - 4(a^2 + a - 3) \geq 0$

and $\quad 3^2 - 2a(3) + a^2 + a - 3 > 0$

$\Rightarrow \quad -a + 3 \geq 0$ and $a^2 - 5a + 6 > 0$

$\Rightarrow \quad a \leq 3$ and $a < 2$ or $a > 3$

$\therefore \quad a < 2$

23 Let $f(x) = x^2 - 2(4k - 1)x + 15k^2 - 2k - 7$, then $f(x) > 0$

$\therefore \quad D < 0$

$\Rightarrow \quad 4(4k - 1)^2 - 4(15k^2 - 2k - 7) < 0$

$\Rightarrow \quad k^2 - 6k + 8 < 0 \Rightarrow 2 < k < 4$

Hence, required integer value of k is 3.

24 Here, $a = 20, d = -\dfrac{2}{3}$

As the common difference is negative, the terms will become negative after some stage. So, the sum is maximum, if only positive terms are added.

Now, $T_n = 20 + (n - 1)\left(-\dfrac{2}{3}\right) \geq 0$

$\Rightarrow \quad 60 - 2(n - 1) \geq 0$

$\Rightarrow \quad 62 \geq 2n \Rightarrow 31 \geq n$

Thus, the first 31 terms are non-negative.

$\therefore$ Maximum sum,

$S_{31} = \dfrac{31}{2}\left[2 \times 20 + (31 - 1)\left(-\dfrac{2}{3}\right)\right]$

$= \dfrac{31}{2}(40 - 20) = 310$

25 First series has common difference 4 and second series has common difference 5.

Hence, the series with common terms has common difference is equal to the LCM of 4 and 5 i.e. 20.

Since, the first common term is 21.

So, the series will be 21, 41, 61, …, 401 which has 20 terms.

26 Given, $a_1 + a_2 + a_3 = 21$

$\Rightarrow \quad a(1 + r + r^2) = 21$

and $\quad a_4 + a_5 + a_6 = 168$

$\Rightarrow \quad ar^3(1 + r + r^2) = 168$

$\therefore \quad r^3 = 8 \Rightarrow r = 2$

and $\quad a(1 + 2 + 4) = 21$

$\therefore \quad a = 3$

27 $T_r = \dfrac{r}{1 + r^2 + r^4}, r = 1, 2, 3, \ldots, n$

$= \dfrac{r}{(r^2 + r + 1)(r^2 - r + 1)}$

$= \dfrac{1}{2}\left[\dfrac{1}{r^2 - r + 1} - \dfrac{1}{r^2 + r + 1}\right]$

$\therefore \sum_{r=1}^{n} T_r = \dfrac{1}{2}\sum_{r=1}^{n}\left[\dfrac{1}{r^2 - r + 1} - \dfrac{1}{r^2 + r + 1}\right]$

$= \dfrac{1}{2}\left[\left(1 - \dfrac{1}{3}\right) + \left(\dfrac{1}{3} - \dfrac{1}{7}\right) + \ldots + \left(\dfrac{1}{n^2 - n + 1} - \dfrac{1}{n^2 + n + 1}\right)\right]$

$= \dfrac{1}{2}\left[1 - \dfrac{1}{n^2 + n + 1}\right] = \dfrac{n^2 + n}{2(n^2 + n + 1)}$

28 Here, X is $n \times 1$, C is $n \times n$ and X' is $1 \times n$ order matrix. Therefore, $X'CX$ is 1×1 order matrix. Let $X'CX = K$

Then, $(X'CX)' = X'C'X''$

$= X'(-C)X = -K$

$\Rightarrow \quad 2K = O$

$\therefore \quad K = O$

29 Since, the determinant of a skew-symmetric matrix of an odd order is zero. Therefore, the matrix is singular.

30 We know that, if A is a square matrix of order n and B is the matrix of cofactors of elements of A. Then,

$$|B| = |A|^{n-1}$$
$$\therefore \Delta = |A|^{3-1} = 5^{3-1} = 25$$

31 Considering CC as single letter, U, CC, E can be arranged in $3!$ ways

Here, $\times$ U $\times$ CC $\times$ E $\times$

Hence, the required number of ways
$$= 3! \cdot {}^4C_3 = 24$$

32 Let $(\sqrt{2}+1)^6 = I + F$, where I is an integer and $0 \le F < 1$

Let $\quad f = (\sqrt{2}-1)^6$

Now, $\sqrt{2}-1 = \dfrac{1}{\sqrt{2}+1}$
$$\Rightarrow \quad 0 < \sqrt{2}-1 < 1$$

Also, $I + F + f = (\sqrt{2}+1)^6 + (\sqrt{2}-1)^6$
$$= 2\,[{}^6C_0 \cdot 2^3 + {}^6C_2 \cdot 2^2 + {}^6C_4 \cdot 2 + {}^6C_6]$$
$$= 2\,(8 + 60 + 30 + 1) = 198$$

Hence, $F + f = 198 - I$ is an integer.

But $\quad 0 < F + f < 2$

$\therefore \quad F + f = 1$ and $I = 197$

33 Given, $\quad B = -A^{-1}BA$
$$\Rightarrow \quad AB = -A\,(A^{-1}BA)$$
$$\Rightarrow \quad AB = -I(BA)$$
$$\therefore AB + BA = O$$

34 Applying $R_3 \to R_3 - pR_1 - R_2$
$$\Rightarrow \begin{vmatrix} xp+y & x & y \\ yp+z & y & z \\ -(xp^2+2yp+z) & 0 & 0 \end{vmatrix} = 0$$
$$\Rightarrow -(xp^2+2yp+z)(xz-y^2) = 0$$

Hence, x, y and z are in GP.

35 For non-trivial solution, we must have
$$\begin{vmatrix} 1 & k & 3 \\ 3 & k & -2 \\ 2 & 3 & -4 \end{vmatrix} = 0$$

Applying

$R_2 \to R_2 - 3R_1, R_3 \to R_3 - 2R_1$,
$$\begin{vmatrix} 1 & k & 3 \\ 0 & -2k & -11 \\ 0 & 3-2k & -10 \end{vmatrix} = 0$$
$$\Rightarrow \quad 20k + 11\,(3-2k) = 0$$
$$\Rightarrow \quad k = \frac{33}{2}$$

36 Atleast one green ball can be selected out of 5 green balls in $2^5 - 1$, i.e. 31 ways. Similarly, atleast one blue ball can be selected from 4 blue balls in $2^4 - 1 = 15$ ways and atleast one red or not red can be selected in $2^3 = 8$ ways.

Hence, the required number of ways $= 31 \times 15 \times 8 = 3720$

37 LHS $= a\,[C_0 - C_1 + C_2 - C_3 + \ldots + (-1)^n\,C_n]$
$$+ [C_1 - 2C_2 + 3C_3 - \ldots + (-1)^{n-1} \cdot nC_n]$$
$$= a \cdot 0 + 0 = 0$$

38 Given, equation is
$$x^3 + ex^2 - ex - e = 0$$

Applying $R_2 \to R_2 - R_1$

and $R_3 \to R_3 - R_2$,
$$\Delta = \begin{vmatrix} (1+\alpha) & 1 & 1 \\ -\alpha & \beta & 0 \\ 0 & -\beta & \gamma \end{vmatrix}$$
$$= (1+\alpha)\,(\beta\gamma - 0) + \alpha(\gamma) + \alpha\beta$$
$$= \alpha\beta + \beta\gamma + \gamma\alpha + \alpha\beta\gamma$$
$$= -e + e = 0$$

39 Now, $2^{4n} = (1+15)^n$
$$= 1 + {}^nC_1 \cdot 15 + {}^nC_2 \cdot 15^2 + {}^nC_3 \cdot 15^3 + \ldots$$
$$\therefore \quad 2^{4n} - 1 - 15n = 15^2\,[{}^nC_2 + {}^nC_3 \cdot 15 + \ldots]$$
$$= 225\,k$$

Hence, it is divisible by 225.

40 $(1-2x)^{-1/2}\,(1-4x)^{-5/2}$
$$= (1+x)\,(1+10x)$$

[neglecting higher power]
$$= 1 + 11x$$

[neglecting higher power]
$$= 1 + kx$$
$$\therefore \quad k = 11$$

41 Applying $R_1 \to R_1 + R_2 + R_3$, we get
$$\begin{vmatrix} 2x+10 & 2x+10 & 2x+10 \\ 2 & 2x & 2 \\ 7 & 6 & 2x \end{vmatrix} = 0$$

Taking $2x + 10$ common from R_1 and applying

$C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$, we get
$$2\,(x+5)\begin{vmatrix} 1 & 0 & 0 \\ 2 & 2x-2 & 0 \\ 7 & -1 & 2x-7 \end{vmatrix} = 0$$
$$\Rightarrow \quad 2\,(x+5)\,(2x-2)\,(2x-7) = 0$$
$$\therefore \quad x = -5, 1, 3.5$$

42 For ascending power of x, we take the expression
$$\left(\frac{2}{3x^2} + 3x\right)^{12}$$
$$\therefore T_8 \text{ in } \left(\frac{2}{3x^2} + 3x\right)^{12}$$
$$= {}^{12}C_7 \left(\frac{2}{3x^2}\right)^{12-7} (3x)^7$$
$$= \frac{12!}{7!\,5!}\left(\frac{2}{3x^2}\right)^5 (3x)^7$$
$$= \frac{12 \times 11 \times 10 \times 9 \times 8}{5 \times 4 \times 3 \times 2} \times \frac{2^5 \times 3^2}{x^3}$$
$$= \frac{228096}{x^3}$$

43 $(3-5x)^{11} = 3^{11}\left(1 - \dfrac{5x}{3}\right)^{11}$
$$= 3^{11}\left(1 - \frac{1}{3}\right)^{11} \qquad \left[\because x = \frac{1}{5}\right]$$
$$\therefore \text{ Greatest term} = \frac{|x|\,(n+1)}{(|x|+1)}$$
$$= \frac{\left|-\frac{1}{3}\right|(11+1)}{\left|-\frac{1}{3}\right| + 1} = 3$$

Now, $T_3 = 3^{11} \cdot {}^{11}C_2 \left(-\dfrac{1}{3}\right)^2$
$$= 3^{11}\left(\frac{11 \cdot 10}{1 \cdot 2} \times \frac{1}{9}\right) = 55 \times 3^9$$

44 Let the number of papers be n.

$\therefore$ Total number of ways to fail or pass
$$= {}^nC_0 + {}^nC_1 + \ldots + {}^nC_n = 2^n$$

$\therefore$ Total number of ways to fail
$$= 2^n - 1$$

[since, there is only one way to pass]

According to the question,
$$2^n - 1 = 63 \Rightarrow 2^n = 2^6 \Rightarrow n = 6$$

45 Let $A = \{a_1, a_2, \ldots, a_n\}, 1 \le i \le n$

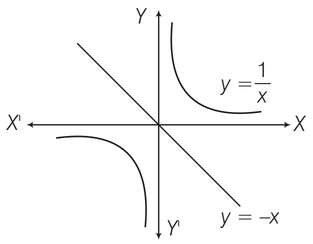
(i) $a_i \in P, a_i \in Q$ $\quad$ (ii) $a_i \notin P, a_i \notin Q$

(iii) $a_i \notin P, a_i \in Q$ $\quad$ (iv) $a_i \in P, a_i \notin Q$

So, $P \cap Q$ contains exactly two elements, taking 2 elements in (i) and $(n-2)$ elements in (ii), (iii) and (iv).

$\therefore$ Required number of ways
$$= {}^nC_2 \times 3^{n-2}$$

46 It is clear from the graph that two curves do not intersect anywhere.
$$\therefore \quad A \cap B = \phi$$



47 $\therefore$ Required number of ways
$$= {}^{16}C_3 - {}^{8}C_3 = 560 - 56 = 504$$

48 Check through options, the condition $2^n > 2n + 1$ is valid for $n \geq 3$.

49 Here, $\alpha + \beta = \sum_{k=1}^{6} w^k = \frac{w(1 - w^6)}{1 - w}$
$$= -1 \qquad [\because w^7 = 1]$$

50 $\because H(x) = \frac{f(x)}{g(x)} = \frac{1 - 2\sin^2 x}{\cos 2x}$
$$= \frac{\cos 2x}{\cos 2x} = 1$$
But $\cos 2x \neq 0$
$\Rightarrow 2x \neq n\pi + \frac{\pi}{2}, n \in I$
$\therefore x \in R \sim \left\{(2n+1)\frac{\pi}{4}, n \in I\right\}$
and range = {1}

51 $T_n = {}^nC_3$ and $T_{n+1} - T_n = 21$
$\Rightarrow {}^{n+1}C_3 - {}^nC_3 = 21$
$\Rightarrow {}^nC_2 + {}^nC_3 - {}^nC_3 = 21$
$\Rightarrow {}^nC_2 = 21$
$\Rightarrow \frac{n(n-1)}{2} = 21$
$\Rightarrow n^2 - n - 42 = 0$
$\Rightarrow (n-7)(n+6) = 0$
$\therefore n = 7 \qquad [\because n \neq -6]$

52 Since, $|r| < 1 \Rightarrow -1 < r < 1$
Also, $a = 5(1 - r) \Rightarrow 0 < a < 10$
[$\because$ at $r = -1, a = 10$
and at $r = 1, a = 0$]

53 The number of divisor of $10^m = 2^m 5^m$ is $(m+1)^2$.
$\therefore$Number of divisors which divide 10^{2009} but not 10^{2008} is
$$(2010)^2 - (2009)^2 = 4019$$

54 If $X = O$, then $X' AX = O \Rightarrow B = O$, a contradiction.
Let $\det(X) = a$, then $\det(X') = a$
$\therefore \det(X' AX) = \det(B)$
$\Rightarrow a(-1)a = -4$
[$\because \det(X'AX) = \det(X')\det(A)\det(X)$]
$\therefore a = \pm 2$
As $\det(X) \neq 0$, X cannot be a singular matrix.

55 $T_3 = {}^mC_2 (2x)^{m-2} \left(\frac{1}{x^2}\right)^2$
$$= {}^mC_2 (2)^{m-2} \cdot x^{m-6}$$
For independent of x, put
$m - 6 = 0 \Rightarrow m = 6$
$\therefore T_3 = {}^6C_2 (2)^{6-2} = 15 \times 16 = 240$
According to the question,
$${}^{30}C_1 x^3 = 240$$
$\Rightarrow x^3 = 8$
$\Rightarrow x = 2$

56 $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
$$= 4 + 8 - n(A \cap B)$$
$$= 12 - n(A \cap B)$$
Since, maximum number of element in $n(A \cap B) = 4$.
$\therefore$ Minimum number of element is
$$n(A \cup B) = 12 - 4 = 8$$

57 If λ is a common root of $ax^2 + bx + c = 0$
and $px + q = 0$, then $a\lambda^2 + b\lambda + c = 0$, $p\lambda + q = 0$ and $p\lambda^2 + q\lambda = 0$
Eliminating λ, we obtained $\Delta = 0$.
For Statement II, expanding Δ along C_1, we obtain
$$aq^2 + p(bq - cp) = 0$$
or $a\left(-\frac{q}{p}\right)^2 + b\left(-\frac{q}{p}\right) + c = 0$
Thus, $ax^2 + bx + c = 0$ and $px + q = 0$ have a common root.

58 I. Matrices P and Y are of the orders $p \times k$ and $3 \times k$, respectively.

Therefore, matrix PY will be defined, if $k = 3$.

Consequently PY will be of the order $p \times k$. Matrices W and Y are of the orders $n \times 3$ and $3 \times k$, respectively.

Since, the number of columns in W is equal to the number of rows in Y, matrix WY is well-defined and is of the order $n \times k$. Matrices PY and WY can be added only when their orders are same.

However, PY is of the order $p \times k$ and WY is of the order $n \times k$, therefore we must have $p = n$. Thus, $k = 3$ and $p = n$ are the restrictions on n, k and p, so that $PY + WY$ will be defined.

II. Matrix X is of the order $2 \times n$.

Therefore, matrix $7X$ is also of the same order.

Matrix Z is of the order $2 \times p$, i.e. $2 \times n$. [$\because n = p$]

Therefore, matrix $5Z$ is also of the same order.

Now, both the matrices $7X$ and $5Z$ are of the order $2 \times n$.

Thus, matrix $7X - 5Z$ is well-defined and is of the order $2 \times n$.